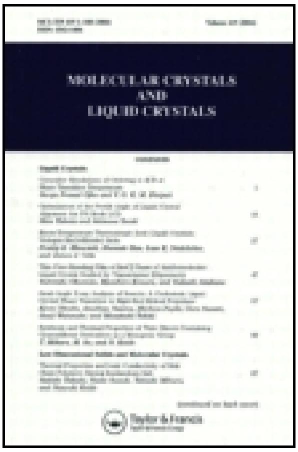

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T-matrix Theory of Light Scattering by Uniformly Anisotropic Spherical Scatterers

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We extend the T-matrix approach to light scattering by spherical particles to the simple case in which the scatterers are optically anisotropic. Specifically, we consider the spherical particles that are uniformly anisotropic. We find that the T-matrix theory can be formulated using a modified T-matrix ansatz with suitably defined modes. We derive these modes by relating the wave packet representation and expansions of electromagnetic field over spherical harmonics. We present preliminary results of numerical calculations of the scattering by spherical droplets. We concentrate on cases in which the scattering is due only to the local optical anisotropy within the scatterer.

Keywords: light scattering; anisotropy; T-matrix theory

1 INTRODUCTION

There are a large number of physical contexts in which it is useful to understand light scattering by impurities [1]. A particular example of recent interest concerns liquid crystal devices. There are now a number of systems in which liquid crystal droplets are suspended in a polymer matrix – the so-called PDLC systems – or the inverse system, involving colloids now with a nematic liquid crystal solvent. These inverse systems are commonly known as filled nematics [2, 3]. In such systems one needs to calculate light scattering by composite anisotropic particles embedded in an isotropic or an anisotropic matrix.

A number of approaches such as the well-known Rayleigh-Gans (RGA) and Anomalous Diffraction Approximations (ADA) are available to study light scattering by complex objects (see, for example, Chap. 2 in [4] and references therein). The T-matrix method was introduced by Waterman [5] and can be regarded as a modern extension of the Mie strategy, dating back almost a century to the classical exact solution due to Mie [6].

The analysis of a Mie-type theory uses a systematic expansion of the electromagnetic field over vector spherical harmonics. The specific form of the expansions is known as the T-matrix ansatz. This has been widely used in the related problem of light scattering by nonspherical particles [4, 7]. The T-matrix approach combines computational efficiency and well defined transformation properties of the spherical harmonics under rotations. But this theory applies to scattering by particles with isotropic dielectric properties.

Recently in [8] we have studied the scattering problem by a uniaxial optically anisotropic annular layer surrounding a spherical particle for a number of optical axis distributions. We have developed the generalised Mie theory as an extension of the T–matrix ansatz [7].

In this paper we explain how this approach can be extended to the case of uniformly anisotropic spherical particles.

The layout of the paper is as follows. General discussion of the model is given in Sec. 2. Then in Sec. 3 we outline the T-matrix formalism for the isotropic medium in the form suitable for subsequent generalisation. In Sec. 4 we describe the method to put the scattering problem into the language of T–matrix by linking the representations of plane wave packets and of spherical harmonics. We also present some numerical results. Finally, in Sec. 5 we draw together the results and make some concluding remarks.

2 MODEL

We consider scattering by a spherical particle of radius R_1 embedded in a uniform isotropic dielectric medium with dielectric constant $\epsilon_{ij} = \epsilon\delta_{ij}$ and magnetic permeability $\mu_{ij} = \mu\delta_{ij}$.

The dielectric tensor $\boldsymbol{\epsilon}$ within the scatterer is locally uniaxial.

The optical axis distribution is defined by the vector field $\hat{\mathbf{n}}$. (Hats will denote unit vectors.) Then within the scatterer $\boldsymbol{\epsilon}_{ij} = \epsilon_{\perp}[\delta_{ij} + u(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})_{ij}]$, where $u = (\epsilon_{||} - \epsilon_{\perp})/\epsilon_{\perp}$ is the *anisotropy parameter*. The unit vector $\hat{\mathbf{n}}$ corresponds to a liquid crystal director for material within the sphere. In this paper we put the magnetic permittivity equal to the unit, so that the refractive indices are $n \equiv \sqrt{\epsilon}$ and $n_o \equiv \sqrt{\epsilon_{\perp}}$.

For uniform anisotropy the optical axis is parallel to the z-axis, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. The scattering problem from spherical uniformly anisotropic particles is not exactly soluble [9] and has been previously studied by using RGA and ADA in [10, 11].

3 T–MATRIX THEORY IN ISOTROPIC MEDIUM

In this section we briefly review the formulation of scattering properties in terms of the T–matrix [4, 12]. The technical details, that have been omitted for brevity, can be found in our preprint [13].

3.1 T-matrix

Starting from expansions over the vector spherical harmonic basis, $\mathbf{Y}_{j+\delta\, j\, m}(\phi, \theta) \equiv \mathbf{Y}_{j+\delta\, j\, m}(\hat{\mathbf{r}})$ $(\delta = 0, \pm 1)$ [14] and using separation of variables, the electromagnetic field in an isotropic medium can be written as a sum of the harmonics:

$$\mathbf{E}_{jm} = \alpha_{jm}\mathbf{M}^{(m)}_{jm}(\rho, \hat{\mathbf{r}}) + \beta_{jm}\tilde{\mathbf{M}}^{(m)}_{jm}(\rho, \hat{\mathbf{r}}) - - n^{-1}\left(\tilde{\alpha}_{jm}\mathbf{M}^{(e)}_{jm}(\rho, \hat{\mathbf{r}}) + \tilde{\beta}_{jm}\tilde{\mathbf{M}}^{(e)}_{jm}(\rho, \hat{\mathbf{r}})\right), \quad (1a)$$

$$\mathbf{H}_{jm} = \tilde{\alpha}_{jm}\mathbf{M}^{(m)}_{jm}(\rho, \hat{\mathbf{r}}) + \tilde{\beta}_{jm}\tilde{\mathbf{M}}^{(m)}_{jm}(\rho, \hat{\mathbf{r}}) + + n\left(\alpha_{jm}\mathbf{M}^{(e)}_{jm}(\rho, \hat{\mathbf{r}}) + \beta_{jm}\tilde{\mathbf{M}}^{(e)}_{jm}(\rho, \hat{\mathbf{r}})\right), \quad (1b)$$

where α_{jm}, $\tilde{\alpha}_{jm}$, β_{jm} and $\tilde{\beta}_{jm}$ are integration constants. The vector functions $\mathbf{M}^{(\alpha)}_{jm}$ and $\tilde{\mathbf{M}}^{(\alpha)}_{jm}$ are expressed in terms of spherical Bessel functions, $j_j(x)$, and spherical Hankel functions, $h^{(1)}_j(x)$, [15] and their derivatives as follows:

$$\mathbf{M}^{(m)}_{jm}(\rho,\hat{\mathbf{r}}) = j_j(\rho)\mathbf{Y}^{(m)}_{jm}(\hat{\mathbf{r}}),$$
$$\mathbf{M}^{(e)}_{jm}(\rho,\hat{\mathbf{r}}) = [j_j(\rho)]'\,\mathbf{Y}^{(e)}_{jm}(\hat{\mathbf{r}}) + \frac{\sqrt{j(j+1)}}{\rho}\,j_j(\rho)\mathbf{Y}^{(o)}_{jm}(\hat{\mathbf{r}}), \quad (2)$$

where $\mathbf{Y}^{(m)}_{jm} = \mathbf{Y}_{j\,j\,m}$ and $\mathbf{Y}^{(e)}_{jm} = [j/(2j+1)]^{1/2}\mathbf{Y}_{j+1\,j\,m} + [(j+1)/(2j+1)]^{1/2}\mathbf{Y}_{j-1\,j\,m}$ are magnetic and electric harmonics, respectively, and $\mathbf{Y}^{(0)}_{jm} = [j/(2j+1)]^{1/2}\mathbf{Y}_{j-1\,j\,m} - [(j+1)/(2j+1)]^{1/2}\mathbf{Y}_{j+1\,j\,m}$ are longitudinal harmonics; $[f(x)]' \equiv x^{-1}\dfrac{\mathrm{d}}{\mathrm{d}x}(xf(x))$ and $\rho = n\omega/cr \equiv kr$. The spherical modes $\tilde{\mathbf{M}}^{(\alpha)}_{jm}$ can be obtained from Eq. (2) by replacing the Bessel functions with the Hankel functions. (In what follows a tilde over vectors and matrices will denote making substitution: $j_j \to h^{(1)}_j$.)

Suppose that a transverse plane wave is incident in the direction specified by an unit vector $\hat{\mathbf{k}}_{inc} \equiv \mathbf{e}_0(\hat{\mathbf{k}}_{inc})$, with $\mathbf{k}_{inc} = k\hat{\mathbf{k}}_{inc}$ and the polarisation vector $\mathbf{E}^{(inc)} = \sum_{\nu=\pm1} E^{(inc)}_{\nu}\mathbf{e}_{\nu}(\hat{\mathbf{k}}_{inc})$, where basis vectors $\mathbf{e}_{\pm1}(\hat{\mathbf{k}}_{inc})$ are perpendicular to $\hat{\mathbf{k}}_{inc}$; $\mathbf{e}_{\nu}(\hat{\mathbf{k}}) = \sum_{\mu=-1}^{1} D^{1}_{\mu\nu}(\hat{\mathbf{k}})\,\mathbf{e}_{\mu}$ and $\mathbf{e}_{\pm1}(\hat{\mathbf{k}}) = \mp(\mathbf{e}_x(\hat{\mathbf{k}}) \pm i\,\mathbf{e}_y(\hat{\mathbf{k}}))/\sqrt{2}$. (The symbol $D^{j}_{mm'}$ stands for the Wigner D-function [14, 16].) Then all the coefficients $\{\beta\}$ are equal to zero. The coefficients $\{\alpha\}$ of the expansion (1) for the plane wave take the form [13]:

$$\alpha^{(inc)}_{jm} = i\alpha_j \sum_{\nu=\pm1} D^{j}_{m\nu}(\hat{\mathbf{k}}_{inc})\nu E^{(inc)}_{\nu} \equiv \alpha^{(inc)}_{jm;\,x}\,E^{(inc)}_{x} + \alpha^{(inc)}_{jm;\,y}\,E^{(inc)}_{y},$$
$$\tilde{\alpha}^{(inc)}_{jm} = n\alpha_j \sum_{\nu=\pm1} D^{j}_{m\nu}(\hat{\mathbf{k}}_{inc})E^{(inc)}_{\nu} \equiv \tilde{\alpha}^{(inc)}_{jm;\,x}\,E^{(inc)}_{x} + \tilde{\alpha}^{(inc)}_{jm;\,y}\,E^{(inc)}_{y}, \quad (3)$$

where $\alpha_j = i^{j+1}[2\pi(2j+1)]^{1/2}$.

For an outgoing wave the coefficients $\{\alpha\}$ are zero as required by the Sommerfeld radiation condition, whereas $\{\beta\} = \{\beta^{(sca)}_{jm}, \tilde{\beta}^{(sca)}_{jm}\}$.

So long as the scattering problem is linear, the coefficients $\beta^{(sca)}_{jm}$ and $\tilde{\beta}^{(sca)}_{jm}$ can be written as linear combinations of $\alpha^{(inc)}_{jm}$ and $\tilde{\alpha}^{(inc)}_{jm}$. The coefficients of these combinations define elements of T-matrix. In our case the cylindrical symmetry of the optical axis distribution causes the T-matrix to be diagonal over azimuthal indices m and m': $T^{nn'}_{jm,j'm'} = \delta_{mm'}\,T^{nn'}_{jj';m}$. Then we can conveniently write the relation using matrix notations:

$$\begin{pmatrix} \beta_{jm}^{(sca)} \\ n^{-1}\,\tilde{\beta}_{jm}^{(sca)} \end{pmatrix} = \sum_{j'} \mathbf{T}_{jj';m} \begin{pmatrix} \alpha_{j'm}^{(inc)} \\ n^{-1}\,\tilde{\alpha}_{j'm}^{(inc)} \end{pmatrix} . \tag{4}$$

3.2 Scattering Amplitude Matrix

In the far field region ($\rho \gg 1$) the scattering amplitude matrix $\mathbf{A}(\hat{\mathbf{k}}_{sca}, \hat{\mathbf{k}}_{inc})$, which relates $\mathbf{E}_{sca}$ and the polarisation vector of the incident wave $\mathbf{E}^{(inc)}$ is defined in the following way [1, 7, 12]:

$$E_{\nu}^{(sca)} \equiv (\mathbf{e}_{\nu}^{*}(\hat{\mathbf{k}}_{sca}), \mathbf{E}_{sca}) = \rho^{-1}\mathrm{e}^{i\rho} \sum_{\nu'=\pm 1} \mathbf{A}_{\nu\nu'}(\hat{\mathbf{k}}_{sca}, \hat{\mathbf{k}}_{inc}) E_{\nu'}^{(inc)} , \tag{5}$$

where an asterisk indicates complex conjugation and $\hat{\mathbf{k}}_{sca} = \hat{\mathbf{r}}$.

Asymptotic behaviour of the Hankel functions [15] can now be used to yield the expression for the scattering amplitude matrix in terms of the T–matrix [13]:

$$\mathbf{A}_{\nu\nu'}(\hat{\mathbf{k}}_{sca}, \hat{\mathbf{k}}_{inc}) = -\frac{i}{2} \sum_{m} \sum_{j,j'} [(2j+1)(2j'+1)]^{1/2} D_{m\nu}^{j\,*}(\hat{\mathbf{k}}_{sca}) \cdot$$
$$\cdot\, D_{m\nu'}^{j'}(\hat{\mathbf{k}}_{inc}) \left[\nu\nu' T_{jj';m}^{11} - i\nu\, T_{jj';m}^{12} + i\nu' T_{jj';m}^{21} + T_{jj';m}^{22} \right] . \tag{6}$$

All scattering properties of the system can be computed from the elements of the scattering amplitude matrix.

Let us write down the result for the scattering efficiency, Q, that is the ratio of the total scattering cross-section, C_{sca}, and area of the particle, $S = \pi R_1^2$,

$$Q = \frac{C_{sca}}{S} = I_{inc}^{-1} \sum_{\alpha=x,y} \sum_{\beta=x,y} Q_{\alpha\beta} E_{\alpha}^{(inc)} E_{\beta}^{(inc)\,*} , \tag{7}$$

$$Q_{\alpha\beta} = (kR_1)^{-2}\pi^{-1} \sum_{jm} \left[\beta_{jm;\,\alpha}^{(sca)} \beta_{jm;\,\beta}^{(sca)\,*} + n^{-2} \tilde{\beta}_{jm;\,\alpha}^{(sca)} \tilde{\beta}_{jm;\,\beta}^{(sca)\,*} \right] , \tag{8}$$

where $I_{inc} = \sum_{\nu=\pm 1} \left| E_{\nu}^{(inc)} \right|^2$.

The total scattering cross-section for the uniformly anisotropic scatterer depends on the angle of incidence θ_{inc} which is the angle between the direction of incidence $\hat{\mathbf{k}}_{inc}$ and the direction of the uniform anisotropy. In addition, the scattering cross-section depends on the polarisation of the incident wave.

4 *T*–MATRIX ANSATZ IN ANISOTROPIC MEDIUM

In Sec. 3 we started from the fields in isotropic medium that were expressed in terms of the modes, $\mathbf{M}_{jm}^{(\alpha)}$ and $\tilde{\mathbf{M}}_{jm}^{(\alpha)}$ (see Eq. (2)). This expression is known as the T-matrix ansatz [4, 7].

The results for electromagnetic field within the radially anisotropic scatterer as they are given in [8] can be written in the form similar to the T-matrix ansatz:

$$\mathbf{E}_{jm} = \alpha_{jm}\mathbf{P}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) + \beta_{jm}\tilde{\mathbf{P}}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) - \\ - n^{-1}\left(\tilde{\alpha}_{jm}\mathbf{P}_{jm}^{(e)}(\rho, \hat{\mathbf{r}}) + \tilde{\beta}_{jm}\tilde{\mathbf{P}}_{jm}^{(e)}(\rho, \hat{\mathbf{r}})\right), \tag{9a}$$

$$\mathbf{H}_{jm} = \tilde{\alpha}_{jm}\mathbf{Q}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) + \tilde{\beta}_{jm}\tilde{\mathbf{Q}}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) + \\ + n\left(\alpha_{jm}\mathbf{Q}_{jm}^{(e)}(\rho, \hat{\mathbf{r}}) + \beta_{jm}\tilde{\mathbf{Q}}_{jm}^{(e)}(\rho, \hat{\mathbf{r}})\right). \tag{9b}$$

In the this section we shall find that the fields in uniformly anisotropic medium can also be written in the form (9). We provide the method of defining modes in a uniformly anisotropic material. However, the expressions for the normal modes in this case will differ from those for radial anisotropy [8, 13]. In both cases these modes are: (a) solutions of the Maxwell's equations and (b) deformations of the isotropic spherical harmonics. The latter condition means that the isotropic modes Eqs. (2) are recovered in the weak anisotropy limit, $u \to 0$.

The electromagnetic field inside the uniformly anisotropic region can easily be described in terms of plane waves. In order to make a connection between plane wave representation and expansions over spherical harmonics we begin with an isotropic medium. The result is a relation between the plane wave packets and the spherical harmonics. The procedure will then be generalised to cover the case of a uniformly anisotropic medium, so as to derive a set of "quasi-spherical" normal modes.

4.1 Spherical modes and plane waves in isotropic media

We start with an electromagnetic wave written as a superposition of plane waves that represents the solution of the Maxwell equations for an isotropic medium.

$$\mathbf{E} = \langle \exp(i\rho\,\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\big[E_x(\hat{\mathbf{k}})\,\mathbf{e}_x(\hat{\mathbf{k}}) + E_y(\hat{\mathbf{k}})\,\mathbf{e}_y(\hat{\mathbf{k}})\big]\rangle_{\hat{\mathbf{k}}}\,, \tag{10a}$$

$$\mathbf{H} = n\,\langle \exp(i\rho\,\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\big[E_x(\hat{\mathbf{k}})\,\mathbf{e}_y(\hat{\mathbf{k}}) - E_y(\hat{\mathbf{k}})\,\mathbf{e}_x(\hat{\mathbf{k}})\big]\rangle_{\hat{\mathbf{k}}}\,, \tag{10b}$$

where $\langle\, f\,\rangle_{\hat{\mathbf{k}}} \equiv \int_0^{2\pi} \mathrm{d}\phi_k \int_0^{\pi} \sin\theta_k \mathrm{d}\theta_k\, f$. Wave-like solutions written in terms of spherical coordinate basis functions are given by Eqs. (1) and (2).

In [13] we have derived expressions connecting the isotropic spherical modes and the vector plane waves occurring in the superposition (10). These relations can be used to write the linear combinations of the modes $\mathbf{M}^{(m)}_{jm}(\rho,\hat{\mathbf{r}})$ and $\mathbf{M}^{(e)}_{jm}(\rho,\hat{\mathbf{r}})$ which enter the electromagnetic field harmonics (1) as a superposition of plane waves:

$$\alpha_{jm}\mathbf{M}^{(m)}_{jm}(\rho,\hat{\mathbf{r}}) - n^{-1}\tilde{\alpha}_{jm}\mathbf{M}^{(e)}_{jm}(\rho,\hat{\mathbf{r}}) = \langle \exp(i\rho\,\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \\ \cdot\,\big[E^{(x)}_{jm}(\hat{\mathbf{k}})\,\mathbf{e}_x(\hat{\mathbf{k}}) + E^{(y)}_{jm}(\hat{\mathbf{k}})\,\mathbf{e}_y(\hat{\mathbf{k}})\big]\rangle_{\hat{\mathbf{k}}}\,, \tag{11a}$$

$$\tilde{\alpha}_{jm}\mathbf{M}^{(m)}_{jm}(\rho,\hat{\mathbf{r}}) + n\,\alpha_{jm}\mathbf{M}^{(e)}_{jm}(\rho,\hat{\mathbf{r}}) = n\,\langle \exp(i\rho\,\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \\ \cdot\,\big[E^{(x)}_{jm}(\hat{\mathbf{k}})\,\mathbf{e}_y(\hat{\mathbf{k}}) - E^{(y)}_{jm}(\hat{\mathbf{k}})\,\mathbf{e}_x(\hat{\mathbf{k}})\big]\rangle_{\hat{\mathbf{k}}}\,, \tag{11b}$$

where

$$(4\pi)^2(1,i)\,E^{(x,y)}_{jm}(\hat{\mathbf{k}}) = i^{-j}[\pi(2j+1)]^{1/2}\cdot \\ \cdot\left\{\alpha_{jm}\,D^{(y,x)\,*}_{jm}(\hat{\mathbf{k}}) - i\,n^{-1}\tilde{\alpha}_{jm}\,D^{(x,y)\,*}_{jm}(\hat{\mathbf{k}})\right\}\,, \tag{12a}$$

$$D^{(x,y)}_{jm}(\hat{\mathbf{k}}) = D^{j}_{m,-1}(\hat{\mathbf{k}}) \mp D^{j}_{m,1}(\hat{\mathbf{k}})\,. \tag{12b}$$

We now sum Eqs. (11) over j and m. This enables the amplitudes $E_x(\hat{\mathbf{k}})$ and $E_y(\hat{\mathbf{k}})$ in Eqs. (10) to be expressed in terms of Wigner D-functions:

$$E_x(\hat{\mathbf{k}}) = \sum_{jm} E^{(x)}_{jm}(\hat{\mathbf{k}})\,, \quad E_y(\hat{\mathbf{k}}) = \sum_{jm} E^{(y)}_{jm}(\hat{\mathbf{k}})\,. \tag{13}$$

This equation defines a basis set in the space of the angular dependent amplitudes. The inverse process uses the expansions (13) to derive the expressions for the spherical modes $\mathbf{M}_{jm}^{(\alpha)}$ and $\tilde{\mathbf{M}}_{jm}^{(\alpha)}$ from superpositions of plane waves (10). The procedure works as follows:

(a) We substitute the expansions of the amplitudes $E_x(\hat{\mathbf{k}})$ and $E_y(\hat{\mathbf{k}})$ from Eqs. (13) into the superpositions (10).

(b) From the expressions for the electric (magnetic) fields we obtain the spherical modes as coefficient functions proportional to α_{jm} and $-n^{-1}\tilde{\alpha}_{jm}$ ($\tilde{\alpha}_{jm}$ and $n\,\alpha_{jm}$).

(c) In order to deduce explicit analytical expressions for the modes, we expand the plane waves over vector spherical functions and then perform integration over the angles ϕ_k and θ_k.

(d) Finally, the modes $\tilde{\mathbf{M}}_{jm}^{(\alpha)}$ are derived from the expressions for $\mathbf{M}_{jm}^{(\alpha)}$ by changing the Bessel functions, $j_j(\rho)$, to the Hankel functions, $h_j^{(1)}(\rho)$.

Note that, if a linear combination of Bessel functions, $j_j(\rho)$, represents a solution of linear homogeneous differential equations (Maxwell equations in our case), then the corresponding linear combination of Hankel functions generates another solution. This remark justifies the last step in the procedure described above.

4.2 Wave functions in an anisotropic medium

We start with the expansion (13), and use it to derive formulae for generalised spherical harmonics in the anisotropic medium. The starting point is the well known result for plane waves [17–19]:

$$\begin{aligned}\mathbf{E} = \langle \exp(i\rho_e \hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) E_x(\hat{\mathbf{k}}) \big[\mathbf{e}_x(\hat{\mathbf{k}}) + \frac{u}{1+u}\sin\theta_k\,\hat{\mathbf{z}}\big] + \\ + \exp(i\rho_o \hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) E_y(\hat{\mathbf{k}})\,\mathbf{e}_y(\hat{\mathbf{k}})\rangle_{\hat{\mathbf{k}}}\,, \quad (14a)\end{aligned}$$

$$\begin{aligned}\mathbf{H} = n_o \langle \exp(i\rho_e \hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\, n_e^{-1}\, E_x(\hat{\mathbf{k}})\mathbf{e}_y(\hat{\mathbf{k}}) - \\ - \exp(i\rho_o \hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) E_y(\hat{\mathbf{k}})\mathbf{e}_x(\hat{\mathbf{k}})\rangle_{\hat{\mathbf{k}}}\,, \quad (14b)\end{aligned}$$

where $n_e^2 \equiv n_e^2(\theta_k) = \dfrac{1+u}{1+u\cos^2\theta_k}$ and $\rho_e \equiv n_e(\theta_k)\,\rho_o$.

We now apply the procedure described at the end of the last section to the plane wave packets (14). This gives a representation of the electromagnetic field in the form of the generalised T-matrix ansatz (9) with the modes defined as follows:

$$\mathbf{P}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) = i^{-j}(4\pi)^{-2}\sqrt{\pi(2j+1)}\,\langle\, D_{jm}^{(y)\,*}(\hat{\mathbf{k}})\exp(i\rho_e\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \cdot\left[\mathbf{e}_x(\hat{\mathbf{k}}) + \frac{u}{1+u}\sin\theta_k\,\hat{\mathbf{z}}\right] - iD_{jm}^{(x)\,*}(\hat{\mathbf{k}})\exp(i\rho_o\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\mathbf{e}_y(\hat{\mathbf{k}})\,\rangle_{\hat{\mathbf{k}}}\,, \quad (15)$$

$$\mathbf{P}_{jm}^{(e)}(\rho, \hat{\mathbf{r}}) = i^{-j}(4\pi)^{-2}\sqrt{\pi(2j+1)}\,\langle\, iD_{jm}^{(x)\,*}(\hat{\mathbf{k}})\,\exp(i\rho_e\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \cdot\left[\mathbf{e}_x(\hat{\mathbf{k}}) + \frac{u}{1+u}\sin\theta_k\,\hat{\mathbf{z}}\right] + D_{jm}^{(y)\,*}(\hat{\mathbf{k}})\exp(i\rho_o\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\mathbf{e}_y(\hat{\mathbf{k}})\,\rangle_{\hat{\mathbf{k}}}\,, \quad (16)$$

$$\mathbf{Q}_{jm}^{(m)}(\rho, \hat{\mathbf{r}}) = i^{-j}(4\pi)^{-2}\sqrt{\pi(2j+1)}\,\langle\, -iD_{jm}^{(x)\,*}(\hat{\mathbf{k}})\,\exp(i\rho_e\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \cdot\, n_e^{-1}\,\mathbf{e}_y(\hat{\mathbf{k}}) + D_{jm}^{(y)\,*}(\hat{\mathbf{k}})\exp(i\rho_o\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\mathbf{e}_x(\hat{\mathbf{k}})\,\rangle_{\hat{\mathbf{k}}}\,, \quad (17)$$

$$\mathbf{Q}_{jm}^{(e)}(\rho, \hat{\mathbf{r}}) = i^{-j}(4\pi)^{-2}\sqrt{\pi(2j+1)}\,\langle\, D_{jm}^{(y)\,*}(\hat{\mathbf{k}})\,\exp(i\rho_e\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\cdot \cdot\, n_e^{-1}\,\mathbf{e}_y(\hat{\mathbf{k}}) + iD_{jm}^{(y)\,*}(\hat{\mathbf{k}})\exp(i\rho_o\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\,\mathbf{e}_x(\hat{\mathbf{k}})\,\rangle_{\hat{\mathbf{k}}}\,, \quad (18)$$

Since the electromagnetic field must be regular at the origin, the harmonics inside the droplet are now given by

$$\mathbf{E}_{jm} = \alpha_{jm}^{(c)}\,\mathbf{P}_{jm}^{(m)}(\rho_o, \hat{\mathbf{r}}) - n_o^{-1}\,\tilde{\alpha}_{jm}^{(c)}\,\mathbf{P}_{jm}^{(e)}(\rho_o, \hat{\mathbf{r}})\,, \quad (19a)$$

$$\mathbf{H}_{jm} = \tilde{\alpha}_{jm}^{(c)}\,\mathbf{Q}_{jm}^{(m)}(\rho_o, \hat{\mathbf{r}}) + n_o\,\alpha_{jm}^{(c)}\,\mathbf{Q}_{jm}^{(e)}(\rho_o, \hat{\mathbf{r}})\,, \quad (19b)$$

where $\rho_o = m_o kr = m_o\rho$ and $m_o \equiv n_o/n$ is the optical contrast.

4.3 Equations for T–Matrix

In order to calculate the elements of T–matrix, we need to use continuity of the tangential components of the electric and magnetic fields as boundary conditions at $r = R_1$. The continuity conditions at the outside of the droplet, $r = R_1$, can then be written in matrix notation as follows:

$$\sum_{j' \geq |m|} \mathbf{R}_1^{jj';m} \begin{pmatrix} \alpha_{j'm}^{(c)} \\ \tilde{\alpha}_{j'm}^{(c)} \end{pmatrix} = \mathbf{\Gamma}_1^{j} \begin{pmatrix} \alpha_{jm}^{(inc)} \\ \tilde{\alpha}_{jm}^{(inc)} \end{pmatrix} + \tilde{\mathbf{\Gamma}}_1^{j} \begin{pmatrix} \beta_{jm}^{(sca)} \\ \tilde{\beta}_{jm}^{(sca)} \end{pmatrix}\,, \quad (20)$$

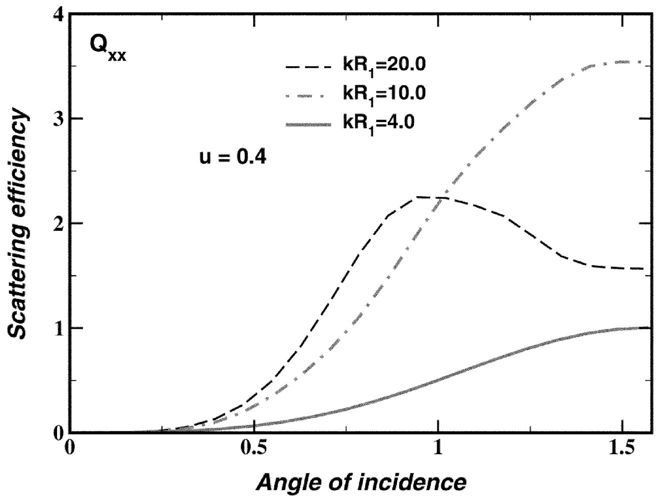

FIGURE 1 Dependence of the scattering efficiency on the angle of incidence for uniformly anisotropic droplet at various values of the size parameter and $u = 0.4$. The refractive indices n and n_o are matched.

$$\mathbf{\Gamma}^{j}(r) = \begin{pmatrix} j_j(\rho) & 0 \\ n[j_j(\rho)]' & 0 \\ 0 & j_j(\rho) \\ 0 & -n^{-1}[j_j(\rho)]' \end{pmatrix}, \tag{21}$$

where the index 1 indicates that matrix elements are calculated at the boundary of droplet, $r = R_1$. For the droplet the matrix $\mathbf{R}^{jj';m}(r)$ takes the form:

$$\mathbf{R}^{jj';m}(r) = \begin{pmatrix} p^{(m,m)}_{jj';m}(\rho_o) & -n_o^{-1}\, p^{(m,e)}_{jj';m}(\rho_o) \\ n_o\, q^{(e,e)}_{jj';m}(\rho_o) & q^{(e,m)}_{jj';m}(\rho_o) \\ n_o\, q^{(m,e)}_{jj';m}(\rho_o) & q^{(m,m)}_{jj';m}(\rho_o) \\ p^{(e,m)}_{jj';m}(\rho_o) & -n_o^{-1}\, p^{(e,e)}_{jj';m}(\rho_o) \end{pmatrix}. \tag{22}$$

Evaluating the coefficient functions $p^{(\beta,\alpha)}_{jj';m}(\rho) = \langle \mathbf{Y}^{(\beta)\,*}_{jm}(\hat{\mathbf{r}})\cdot\mathbf{P}^{(\alpha)}_{j'm}(\rho,\hat{\mathbf{r}})\rangle_{\hat{\mathbf{r}}}$ and $q^{(\beta,\alpha)}_{jj';m}(\rho) = \langle \mathbf{Y}^{(\beta)\,*}_{jm}(\hat{\mathbf{r}})\cdot\mathbf{Q}^{(\alpha)}_{j'm}(\rho,\hat{\mathbf{r}})\rangle_{\hat{\mathbf{r}}}$ involves computing some

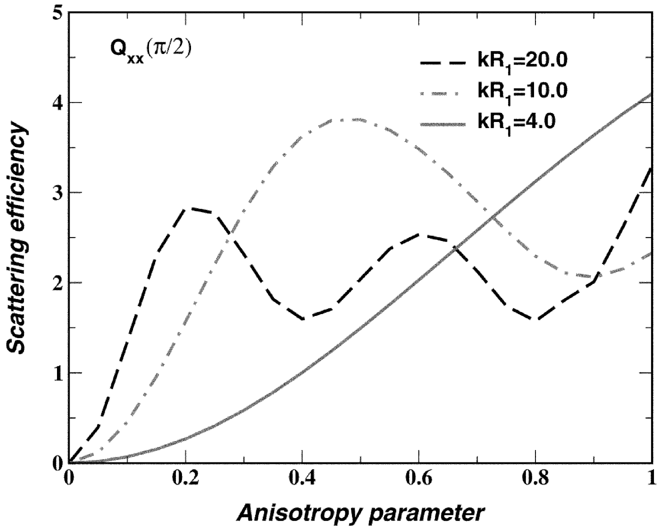

FIGURE 2 Scattering efficiencies of uniformly anisotropic droplets versus the anisotropy parameter at various values of the size parameter for $\theta_{inc} = \pi/2$ and $n = n_o$.

products of Bessel spherical function and Wigner D–functions and integrating these expressions over θ_k. Their numerical evaluation is relatively easy.

4.4 Numerical Results

For uniformly anisotropic droplets T-matrix can only be computed numerically by solving the system of equations (20). In this section we present some numerical results for the scattering efficiency defined by Eqs. (7) and (8). We are primarily interested in anisotropy-induced scattering. In order to concentrate on this test case, we consider the case when the refractive indices n and n_o are equal.

When the refractive indices are matched, $n = n_o$, it is expected that the scatterer does not change the y component of the incident wave, which simply transforms into the ordinary wave inside the droplet without being affected by the scattering process. It can be shown that the system (20) is consistent with this conclusion.

The dependence of the scattering efficiency on the angle of incidence is shown in Fig. 1. If the size parameter, kR_1, is not very large, the scattering efficiency Q_{xx} is a monotonically increasing function of the angle of incidence, θ_{inc}, in the region from 0 to $\pi/2$. By symmetry $Q_{xx}(\theta_{inc}) = Q_{xx}(\pi/2 - \theta_{inc})$, and so the scattering efficiency decreases in the range from $\pi/2$ to π.

In Fig. 1 we show what happens for shorter wavelength and thus higher values of kR_1. Now, for for relatively large values of the size parameter, the cross-section dependence on the angle is incidence is no longer monotonic. For example, at $kR_1 = 20.0$, the angle at which the scattering efficiency Q_{xx} reaches its maximum value is no longer at $\pi/2$.

The scattering efficiencies as a function of the anisotropy parameter, $0 \leq u \leq 1$, at different values of the size parameter are plotted in Fig. 2. It is seen that an increase in the size parameter leads to the appearance of peaks in this range of u.

5 DISCUSSION AND CONCLUSIONS

In this paper we have developed a T-matrix approach which can describe light scattering by spherical scatterers containing uniformly anisotropic material. In this case the light-scattering problem is not exactly soluble. The key point is that the *exact solutions* for uniformly anisotropic medium are known as plane waves, whereas the spherical shape of the particle requires using some kind of spherical modes.

We have found that, by choosing the appropriate basis in $\hat{\mathbf{k}}$–space, we can define 'quasi-spherical' normal modes. These modes are *exact solutions* of Maxwell's equations and as such mix different angular momentum. However, in the limit of zero anisotropy, these modes tend to familiar spherical modes. More importantly, these quasi-spherical modes turn out to be relatively easily accessible computationally. Thus, there is every reason to suppose that the strategy can be adopted in rather more complicated situations.

One such problem is the light scattering problem for a Faraday-active sphere. This problem has been treated using perturbation theory in [20] to explain the origin of magneto-transverse light diffusion known as the "photonic Hall effect" [21, 22].

The T–matrix formalism is a natural language within which to discuss more general multiple scattering processes. Beginning with single scattering theories of the type discussed in this paper, one can in principle construct an effective medium theory using, for example, the coherent potential approximation (CPA) or coated CPA [23, 24]. These theories determine effective optical characteristics of the medium from the condition that the scattering cross section is minimal or equal to zero on average. Since this requires averaging over director orientations, it is important to use basis functions with well defined transformation properties under rotations.

In order to show that the T-matrix approach can be used in an efficient numerical treatment of the scattering problem, we have also presented some of the numerical results calculated using the T-matrix theory. In particular, we have studied the scattering efficiency of uniformly anisotropic droplets in which the ordinary refractive index matches the refractive index of the material surrounding them.

The results of this work can be regarded as the first step towards more comprehensive study of light scattering by anisotropic scatterers.

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References

[1.] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978).
[2.] M. Kreuzer and R. Eidenschink, in *Liquid Crystals in Complex Geometries*, edited by G. Crawford and S. Žumer (Taylor & Francis, London, 1996), chap. 15.
[3.] T. Bellini, N. Clark, V. Degiorgio, F. Mantegazza, and G. Natale, *Phys. Rev. E* **57**, 2996 (1998).
[4.] M. Mishchenko, J. Hovenier, and L. Travis, eds., *Light Scattering by Nonspherical Particles: Theory, Measurements and Applications* (Academic Press, New York, 2000).
[5.] P. Waterman, *Phys. Rev. D* **3**, 825 (1971).
[6.] G. Mie, *Ann. Phys. (Leipzig)* **25**, 377 (1908).

[7.] M. Mishchenko, L. Travis, and D. Mackowski, *J. of Quant. Spectr. & Radiat. Transf.* **55**, 535 (1996).
[8.] A. Kiselev, V. Reshetnyak, and T. Sluckin, *Opt. Spectrosc.* **89**, 907 (2000).
[9.] C. Boren and D. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley-Interscience, New York, 1983).
[10.] S. Žumer and J. Doane, *Phys. Rev. A* **34**, 3373 (1986).
[11.] S. Žumer, *Phys. Rev. A* **37**, 4006 (1988).
[12.] R. Newton, *Scattering Theory of Waves and Particles* (Springer, Heidelberg, 1982), 2nd ed.
[13.] A. Kiselev, V. Reshetnyak, and T. Sluckin, `physics/0109022`, (2001).
[14.] L. Biedenharn and J. Louck, *Angular Momentum in Quantum Physics* (Addison–Wesley, Reading, Massachusetts, 1981).
[15.] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1972).
[16.] I. Gelfand, R. Minlos, and Z. Shapiro, *Representations of Rotation and Lorenz Groups and Their Applications* (Pergamon Press, Oxford, 1963).
[17.] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1980), 2nd ed.
[18.] L. Landau and E. Lifshitz, *Elecrtodynamics of Continuous Media* (Pergamon, Oxford, 1984).
[19.] H. Stark and T. Lubensky, *Phys. Rev. E* **55**, 514 (1997).
[20.] D. Lacoste, B. van Tiggelen, G. Rikken, and A. Sparenberg, *J. Opt. Soc. Am. A* **15**, 1636 (1998).
[21.] B. van Tiggelen, *Phys. Rev. Lett.* **75**, 422 (1995).
[22.] G. Rikken and B. van Tiggelen, *Nature* **381**, 54 (1996).
[23.] X. Jing, P. Sheng, and M. Zhou, *Phys. Rev. A* **46**, 6513 (1992).
[24.] C. Soukoulis, S. Datta, and E. Economou, *Phys. Rev. B* **49**, 3800 (1994).